# Train Tracks, Orbigraphs and CAT(0) Free-by-cyclic Groups

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# Gersten's Theorem

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But $F_3 \rtimes_\Psi \mathbb{Z} \leq \mathrm{Aut}(F_3) \leq \mathrm{Out}(F_4)$. □

# The Free Coxeter Group

Let $W_n = \underbrace{\mathbb{Z}/2\mathbb{Z} * \cdots * \mathbb{Z}/2\mathbb{Z}}_{n \text{ terms}} = \langle a_1, \ldots, a_n \mid a_i^2 \rangle$.

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$$\chi_{ij} \begin{cases} a_j \mapsto a_i a_j a_i \\ a_k \mapsto a_k & k \neq j \end{cases} \qquad \tau_{ij} \begin{cases} a_i \mapsto a_j \\ a_j \mapsto a_i \\ a_k \mapsto a_k & k \neq i, j \end{cases}$$

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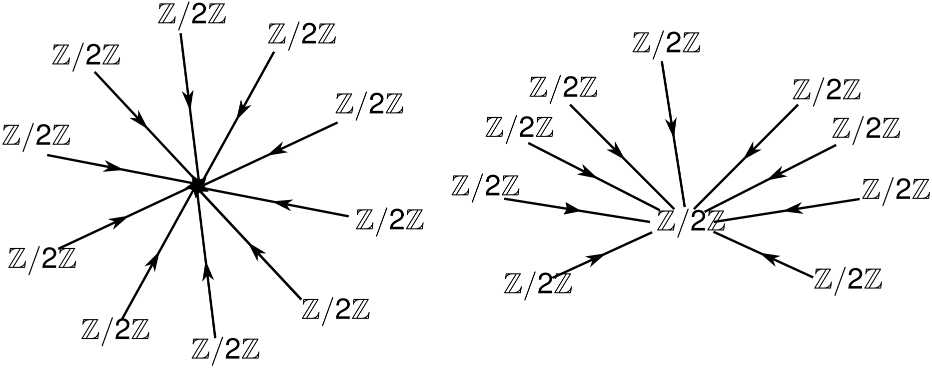
**Question**

*Is* $\mathrm{Out}(W_n)$ *a* $\mathrm{CAT}(0)$ *group?*

# A Topological Model

An ($W_n$-)*orbigraph* $G$ is the quotient of a tree $\widetilde{G}$ by a geometric action of $W_n$ without edge stabilizers.



# The Main Tool

**Theorem (L, '19)**

*Every* $\varphi \in \mathrm{Out}(W)$ *is represented by a relative train track map, a homotopy equivalence* $f \colon G \to G$ *of an orbigraph* $G$ *with nice properties.*

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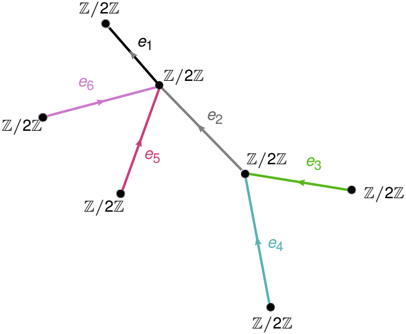
This is a normal form like the Jordan or Nielsen–Thurston normal form. This extends work of Bestvina, Feighn and Handel.

# An Example




$$f \begin{cases} e_1 \mapsto e_1 \\ e_2 \mapsto e_2 . \hat{e}_1 \\ e_3 \mapsto e_4 \\ e_4 \mapsto e_3 e_2 \hat{e}_5 \bar{e}_2 \hat{e}_4 \\ e_5 \mapsto e_6 \\ e_6 \mapsto e_5 \bar{e}_2 \hat{e}_4 e_2 \hat{e}_6 \end{cases}$$

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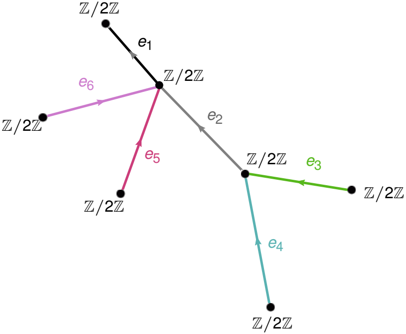


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$$M = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 2 & 0 & 2 \\ & & 0 & 1 & 0 & 0 \\ & & 1 & 2 & 0 & 2 \\ & & 0 & 2 & 0 & 1 \\ & & 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\lambda_1 = \lambda_2 = 1, \ \lambda_3 \approx 3.38\ldots$$

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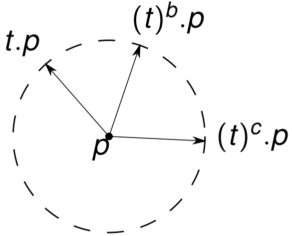


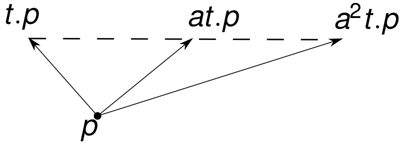

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Use Bridson–Haefliger combination theorem for HNN extensions of $\mathsf{CAT}(0)$ groups.

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