Recognizing Pseudo-Anosov Braids in $\mathrm{Out}(W_n)$

Rylee Lyman, Tufts University

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What is $\mathrm{Out}(W_n)$?

The free Coxeter group of rank n:

$$W_n = (\mathbb{Z}/2\mathbb{Z})^{*n} = \langle a_1, \ldots, a_n \mid a_i^2 = 1 \rangle.$$

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“Nielsen-like” generators:

$$\tau_{ij} \begin{cases} a_i \mapsto a_j \\ a_j \mapsto a_i \\ a_k \mapsto a_k & k \neq i, j \end{cases} \qquad \chi_{ij} \begin{cases} a_j \mapsto a_i a_j a_i \\ a_k \mapsto a_k & k \neq j. \end{cases}$$

A Classification Theorem

Theorem (L, '19)

*Every outer automorphism $\varphi \in \mathrm{Out}(W_n)$ may be represented by a homotopy equivalence $f\colon G \to G$ of a W_n-**orbigraph** with special properties called a **relative train track map.***
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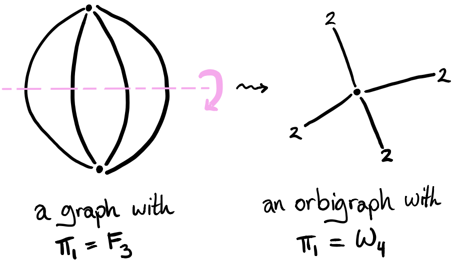
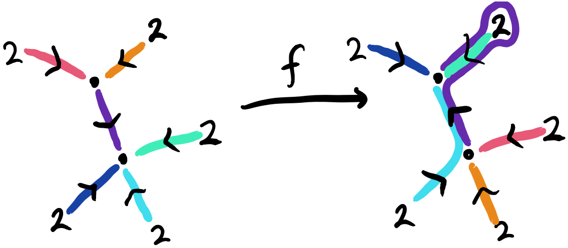
Builds on work of Bestvina, Feighn and Handel for $\mathrm{Out}(F_n)$.

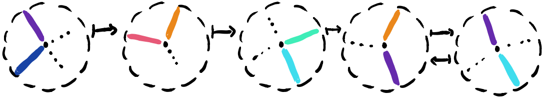


A Train Track Map

A homotopy equivalence $f\colon G \to G$ is a **train track map** when for each edge $e \in G$, the kth iterate $f^k|_e$ is an immersion for all $k \geq 1$.



This is a train track map!



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If $\varphi \in \text{Out}(F_n)$ *is fully irreducible, it is either* ***hyperbolic*** *or* φ^k *can be represented as a pseudo-Anosov homeomorphism of a surface with one boundary component for some* $k \geq 1$*.*

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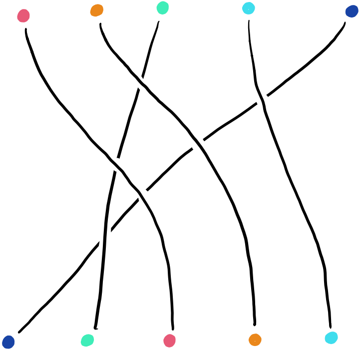
Theorem (L, In Progress)

If $\varphi \in \mathrm{Out}(W_n)$ *is fully irreducible, it is either* ***hyperbolic*** *or* φ^k *can be represented as a pseudo-Anosov braid on an orbifold with one boundary component with orbifold fundamental group* W_n *for some* $k \geq 1$*.*

Braids As Mapping Classes



$\sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_1 \sigma_2$

$$S^2 \setminus \{\infty\}$$

$$\downarrow \; ?$$

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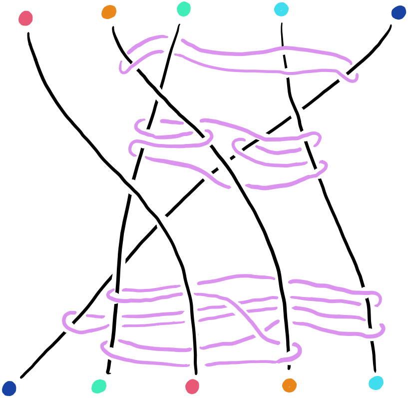
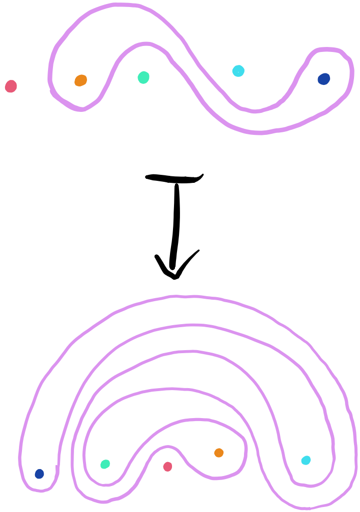
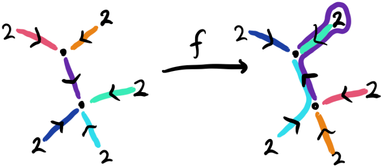
Following A Curve



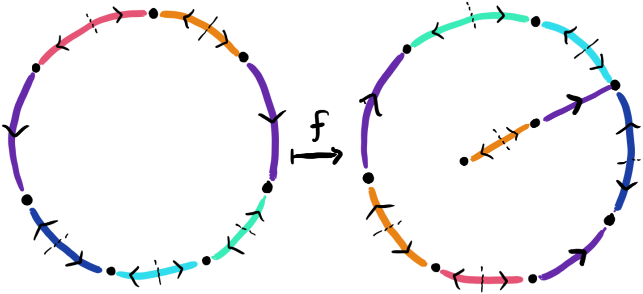
$\sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_1 \sigma_2$



The Example



Need a 2-cell attaching map:



$$\ell \simeq f(\ell)$$