Notes on Orbifolds I

Why is an Étale Lie Groupoid?

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Abstract

The purpose of these notes is to try and understand Haefliger's work on orbifolds. Surely nothing here is original. Indeed, some of it is literally just a restatement of chapter III.$\mathcal{G}$ of [BH99].

1 Basic Definitions

The following is just a restatement of the definition appearing in [BH99]. Let Q be a Hausdorff topological space. A *differentiable orbifold structure of dimension* n on Q is the following data:

1. An open covering $\{V_i : i \in I\}$ of Q.
2. For each $i \in I$, a simply-connected n-manifold X_i and a finite subgroup Γ_i of the group of diffeomorphisms of X_i together with a continuous map $q_i : X_i \to V_i$, called a *uniformizing chart,* such that q_i induces a homeomorphism $\Gamma_i \backslash X_i \to V_i$.
3. For each $x_i \in X_i$ and $x_j \in X_j$ satisfying $q_i(x_i) = q_j(x_j)$, a diffeomorphism h from a connected neighborhood W of x_i to a neighborhood of x_j such that $q_j h = q_i|_W$. Such a map h is called a *change of chart* and is well-defined up to composition with an element of Γ_j. In particular, one verifies that if $i = j$, then h is the restriction of an element of Γ_i.

We call the family $(X_i, q_i)_{i \in I}$ an *atlas* for the orbifold structure on Q. We say two atlases are *equivalent* if their disjoint union satisfies condition (3) above.

We are about to abstract the setting significantly. The idea is that we will use the structure of the atlas in order to define how to work with Q. One goal of these notes is to learn how to work with Q more directly while still producing correct results.

One convenient fact about the category of differentiable manifolds is that it has coproducts even though it does not have all colimits. In other words, from the (X_i, q_i) we can form their disjoint union $X = \coprod_{i \in I} X_i$, which comes with a map $q = \coprod_{i \in I} q_i$ to Q. The changes of charts assemble into a *pseudogroup* of local diffeomorphisms $\mathcal{H}$. To wit, an element of $\mathcal{H}$ is a diffeomorphism defined on an open subset U of X such that $qh = q|_U$, and this collection satisfies the following properties.

1. If $h\colon U \to V$ and $h'\colon U' \to V'$ belong to $\mathcal{H}$, then their composition
$$hh'\colon h'^{-1}(U \cap V') \to h(U \cap V')$$
belongs to $\mathcal{H}$.

2. The restriction of $h \in \mathcal{H}$ to any open set of X belongs to $\mathcal{H}$.

3. The identity $1_X\colon X \to X$ belongs to $\mathcal{H}$.

4. Let $U \subset X$ be an open subset and let $h\colon U \to V$ be a diffeomorphism. If $\{U_j\colon j \in J\}$ is an open covering of U such that there exist elements $h_j \in \mathcal{H}$ such that $h = \bigcup_{j \in J} h_j$, then $h \in \mathcal{H}$. (This is a restricted gluing-type construction.)

We say two points x and x' in X are in the same $\mathcal{H}$-orbit if there exists $h \in \mathcal{H}$ such that $h(x) = x'$. This defines an equivalence relation on X, and the map $q\colon X \to Q$ defines a homeomorphism $\mathcal{H} \backslash X \to Q$. The upshot of this definition is that the pair $(\mathcal{H}, X)$ caaptures all of the data of the orbifold structure on Q, at least up to a notion of equivalence. To flesh this out further, we need one more definition. I will write it out a little pedantically because I am unused to it.

Spaces of Germs Given a pair of spaces X and Y, we can construct a space of *germs* of (partially defined) maps from X to Y in the following way. Points of the space of germs are equivalence classes of pairs (x, f), where x is a point of X and $f\colon U \to Y$ is a map defined on a neighborhood of x. The pair (x, f) is equivalent to (x', f') if $x = x'$ and if f and f' agree when restricted to an open neighborhood of x. Let us call the space of germs $\mathcal{G}$. The definition comes equipped with maps $\alpha\colon \mathcal{G} \to X$ and $\omega\colon \mathcal{G} \to Y$, where $\alpha(x, f) = x$, and $\omega(x, f) = f(x)$.

A basis for the topology on $\mathcal{G}$ is defined in the following way: for U an open subset of X, and $f\colon U \to Y$ a continuous map, the basis element U_f is the union of the germs (x, f) as x varies over the points of U. Thus $\mathcal{G}$ is a Hausdorff space if an only if for each pair (x, f) and (x', f') of distinct germs, we can find open neighborhoods U of x and U' of x' such that the germs of f and f' remain distinct on each point of $U \cap U'$. The map α is *étale*, that is, locally a homeomorphism.

In the case we are interested in, X is the space constructed from our atlas for the orbifold Q, and $\mathcal{G}$ is the space of germs of changes of charts, in other words, $X = Y$, and the maps f we are interested are those diffeomorphisms of open subsets of X satisfying $qf = q$. In this case, the map $\omega\colon \mathcal{G} \to X$ is *also* étale. Note that because $Q = \mathcal{H} \backslash X$ is assumed to be Hausdorff, we have that $\mathcal{G}$ is Hausdorff as well. In fact, the smooth structure on X pulls back to give a smooth structure on $\mathcal{G}$. I believe that if the original open covering I of Q was assumed to be countable, then $\mathcal{G}$ will be second countable. Since this assumption does not appear to be invariant under equivalence of atlases, we won't make this assumption.

Internal Groupoids Recall that a *groupoid* is a category in which every arrow is an isomorphism. We want a definition of a groupoid internal to the category of differentiable manifolds. Thus our definition will be somewhat tailored to that situation. An *étale Lie groupoid* is a pair $(\mathcal{G}, X)$ of differentiable manifolds with a bevy of structure maps between them. The idea is that X is the space of *objects* of our groupoid (in our case, they are also the points of the differentiable manifold X) and $\mathcal{G}$ is the space of *arrows.* (In our case, the space of germs of elements of the pseudogroup $\mathcal{H}$ of changes of charts.) Arrows in a category have a *source* and a *target,* which can be expressed as maps $\mathcal{G} \to X$. We will call the source map $\alpha\colon \mathcal{G} \to X$ and the target map $\omega\colon \mathcal{G} \to X$, and we will require that these maps are *étale,* which just says that they are local homeomorphisms. All of this is already satisfied by the $\mathcal{G}$ that we have been working with. This condition ensures that the following pullback exists

$$\begin{tikzcd} \mathcal{G} \times_X \mathcal{G} \arrow[r, "p_2"] \arrow[d, "p_1"] & \mathcal{G} \arrow[d, "\alpha"] \\ \mathcal{G} \arrow[r, "\omega"] & X \end{tikzcd}$$

Concretely, $\mathcal{G} \times_X \mathcal{G}$ is the space of pairs $(f, g) \in \mathcal{G} \times \mathcal{G}$ such that $\omega(f) = \alpha(g)$. In addition to α and ω, we have three more maps, and diagrams prescribing their behavior. In words, we want to be able to compose pairs of arrows lying in $\mathcal{G} \times_X \mathcal{G}$, and this composition should be associative. For each object in X, there should be a corresponding identity arrow in $\mathcal{G}$, and this arrow should genuinely be the identity for composition. Finally, every arrow should be invertible. To this end, we define three maps: $m\colon \mathcal{G} \times_X \mathcal{G} \to \mathcal{G}$, ${}^{-1}\colon \mathcal{G} \to \mathcal{G}$, and $e\colon X \to \mathcal{G}$. The map m represents composition of arrows. Thus it satisfies the following properties.

1. The source and target of the composite are defined by the following diagrams. The left-hand diagram says that if the composite gf is defined, then $\alpha(gf) = \alpha(f)$, and the right-hand diagram says that in the above situation, we have $\omega(gf) = \omega(g)$

$$\begin{tikzcd} \mathcal{G} \times_X \mathcal{G} \arrow[r, "m"] \arrow[d, "p_1"] & \mathcal{G} \arrow[d, "\alpha"] \\ \mathcal{G} \arrow[r, "\alpha"] & X \end{tikzcd} \qquad \begin{tikzcd} \mathcal{G} \times_X \mathcal{G} \arrow[r, "m"] \arrow[d, "p_2"] & \mathcal{G} \arrow[d, "\omega"] \\ \mathcal{G} \arrow[r, "\omega"] & X. \end{tikzcd}$$

2. The source and target of the identity arrow $1_x = e(x)$ should be x

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[dr, "1_X"] & \mathcal{G} \arrow[d, "\alpha"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "e"] \arrow[dr, "1_X"] & \mathcal{G} \arrow[d, "\omega"] \\ & X. \end{tikzcd}$$

   In particular, these diagrams imply that $e\colon X \to \mathcal{G}$ is an embedding.

3. The inversion map ${}^{-1}\colon \mathcal{G} \to \mathcal{G}$ should swap the source and target of each arrow

$$\begin{tikzcd} \mathcal{G} \arrow[r, "^{-1}"] \arrow[dr, "\omega"'] & \mathcal{G} \arrow[d, "\alpha"] \\ & \mathcal{G} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{G} \arrow[r, "^{-1}"] \arrow[dr, "\alpha"'] & \mathcal{G} \arrow[d, "\omega"] \\ & \mathcal{G}, \end{tikzcd}$$

and the composition of an arrow f with ${}^{-1}(f) = f^{-1}$ should be the identity. Here there are really two diagrams, the second using ω rather than α, but we will draw just the one

$$\begin{tikzcd} \mathcal{G} \arrow[r, "1_{\mathcal{G}} \times {}^{-1}"] \arrow[d, "\alpha"] & \mathcal{G} \times_X \mathcal{G} \arrow[d, "m"] \\ X \arrow[r, "e"] & \mathcal{G}. \end{tikzcd}$$

Here the map $1_{\mathcal{G}} \times {}^{-1}$ sends an arrow f to the pair (f, f^{-1}).

Aside from these, there should be diagrams stipulating that composition of arrows is associative, and that the appropriate identity arrows really behave as identities for composition. We will take all of these things for granted.

Notice that to an étale topological groupoid we can associate a pseudogroup of local homeomorphisms of X in the following way. Elements of the pseudogroup $\mathcal{H}_{\mathcal{G}}$ are homeomorphisms of the form $\omega \circ s$, where $s\colon U \to \mathcal{G}$ is a section of α defined on an open subset U of X. We say the groupoid $(\mathcal{G}, X)$ is a Lie groupoid when elements of the associated pseudogroup $\mathcal{H}_{\mathcal{G}}$ are diffeomorphisms. In fact, in our case it turns out that we have $\mathcal{H} = \mathcal{H}_{\mathcal{G}}$.

Two more bits of notation: given $x \in X$, we define the *isotropy group* at x to be the set

$$\mathcal{G}_x = \{f \in \mathcal{G} : \alpha(f) = \omega(f) = x\}.$$

Notice that the multiplication and inversion on $\mathcal{G}$ send $\mathcal{G}_x$ to itself and induce a group structure on this set, hence our terminology. In our setting, each isotropy group is isomorphic to a subgroup of some finite group Γ_i from our original definition of an orbifold atlas. We also have the *orbit* of x, which is the set

$$\mathcal{G}.x = \{y \in X : \exists f \in \mathcal{G} \text{ such that } \alpha(f) = x \text{ and } \omega(f) = y\}.$$

The orbits assemble into a *quotient space* $\mathcal{G}\backslash X$, which in our case is naturally homeomorphic to Q.

2 Morphisms of Groupoids

Since we have constructed a differentiable groupoid as a stand-in for our orbifold Q, one might expect that maps of orbifolds should be functors of the associated groupoid. This is mostly the case, with one pesky wrinkle. A *homomorphism* of differentiable groupoids $f\colon (\mathcal{G}, X) \to (\mathcal{G}', X')$ is a continuous map $f\colon \mathcal{G} \to \mathcal{G}'$

and a continuous map $f_X\colon X \to X'$ that commutes with the following structure maps:

$$\begin{array}{cccc}
\begin{array}{ccc} X & \xrightarrow{f_X} & X' \\ \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e} \\ \mathcal{G} & \xrightarrow{f} & \mathcal{G}' \end{array} &
\begin{array}{ccc} \mathcal{G} & \xrightarrow{f} & \mathcal{G}' \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} \\ X & \xrightarrow{f_X} & X' \end{array} &
\begin{array}{ccc} \mathcal{G} & \xrightarrow{f} & \mathcal{G}' \\ \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \omega} \\ X & \xrightarrow{f_X} & X' \end{array} &
\begin{array}{ccc} \mathcal{G} & \xrightarrow{f} & \mathcal{G}' \\ \downarrow{\scriptstyle -1} & & \downarrow{\scriptstyle -1} \\ \mathcal{G} & \xrightarrow{f} & \mathcal{G}' \end{array}
\end{array}$$

$$\begin{array}{ccc} \mathcal{G} \times_X \mathcal{G} & \xrightarrow{f \times_X f} & \mathcal{G}' \times_{X'} \mathcal{G}' \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} \\ \mathcal{G} & \xrightarrow{\quad f \quad} & \mathcal{G}'. \end{array}$$

Notice that one or both of the maps involving e and f_X uniquely determine f_X from f. If f_X is a differentiable map, then we say f is a *differentiable homomorphism.* Observe that by definition, f induces a map $\mathcal{G}\backslash f\colon Q \to Q'$. Furthermore, for every point $x \in X$, f induces a homomorphism $f_x\colon \mathcal{G}_x \to \mathcal{G}'_{f_X(x)}$.

A functor of differentiable groupoids is an *equivalence* if it is an equivalence of categories. In practical terms, this means that $\mathcal{G}\backslash f\colon Q \to Q'$ is a homeomorphism and $f_x\colon \mathcal{G}_x \to \mathcal{G}_{f(x)}$ is an isomorphism for all $x \in X$. Notice that f does not have to be a homeomorphism of $\mathcal{G}$, nor even injective!

Here is where we have to pay the piper. Because the atlas X is not an intrinsic part of the definition of the orbifold structure on Q, neither is $\mathcal{G}$, only its equivalence class. Therefore, we will need to consider *equivalence classes* of differentiable groupoids representing our orbifold Q. Hopefully this headache will become easier to manage when we return to the ground in the next section. One helpful fact that we will not attempt to prove is that $(\mathcal{G}, X)$ is equivalent to $(\mathcal{G}', X')$ if and only if there exists a third differentiable groupoid $(\mathcal{G}'', X'')$ and equivalences $(\mathcal{G}'', X'') \to (\mathcal{G}, X)$ and $(\mathcal{G}'', X'') \to (\mathcal{G}', X')$.

3 Morphisms of Orbifolds

In this section we are ready to return to earth with our hard-won insights—I hope. Suppose for a moment that M and N are differentiable manifolds. A continuous map $f\colon M \to N$ of the underlying topological spaces is *differentiable* exactly when, for every pair of charts $q_i\colon X_i \to M$ and $p_j\colon Y_j \to N$, the resulting map $p_j^{-1} \circ f \circ q_i\colon X_i \to Y_j$ is differentiable as a map between Euclidean spaces whenever $f|_{q_i(X_i)} \cap Y_j \neq \varnothing$. In fact, if (X, q) and (Y, p) denote the disjoint unions resulting from our choices of atlas, we would like f to induce a differentiable map $F\colon X \to Y$ such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ M & \xrightarrow{f} & N. \end{array}$$

We can show that this is in fact the case, but it doesn't quite come for free. In general we will have to pass to a finer atlas of M in order to define F. Consider the covering of M given by $f^{-1}(p_j(Y_j))$. Define a new atlas (X_i', q_i') such that for each i, $q_i'(X_i')$ is contained in exactly one of the sets $f^{-1}(p_j(Y_j))$. As before, we assume each of the X_i' are simply connected open subsets of $\mathbb{R}^m$. Now for each pair (i, j), there exists a map $f_{ij}\colon X_i' \to Y_j$ such that the following diagram commutes

$$\begin{tikzcd} X_i' \arrow[r, "f_{ij}"] \arrow[d, "q_i'"] & Y_j \arrow[d, "p_j"] \\ M \arrow[r, "f"] & N, \end{tikzcd}$$

and moreover these maps assemble into a differentiable map $F\colon X' \to Y$ fitting into the commutative diagram we wanted:

$$\begin{tikzcd} X' \arrow[r, "F"] \arrow[d, "q'"] & Y \arrow[d, "p"] \\ M \arrow[r, "f"] & N. \end{tikzcd}$$

Note that if we want, we could choose the atlas (X', q') to *refine* the open covering given by the original atlas (X, q). In this case there is a morphism above the identity of M that expresses the fact that the atlas (X', q') refines (X, q).

Note further that the map F is *equivariant* in the following sense. Let $\mathcal{H}_M$ and $\mathcal{H}_N$ denote the associated pseudogroups of changes of charts. Suppose $h\colon X_i' \to X_j'$ is an element of $\mathcal{H}_M$. Then there exists a $k \in \mathcal{H}_N$ such that $Fh = kF|_{X_i'}$.

This allows us to define an action of F on the corresponding groupoids of germs $\mathcal{G}_{X'}$ and $\mathcal{G}_Y$! To wit, suppose $(x, h) \in \mathcal{G}_{X'}$ represents the germ of an element $h \in \mathcal{H}_M$ at $x \in X'$. We shall use the equation

$$Fh = kF|_{X_i'}$$

to define $F_*\colon \mathcal{G}_{X'} \to \mathcal{G}_Y$ by setting $F_*(x, h) = (F(x), k)$. One of course has to check that the choice of k is well-defined on the level of germs. In fact, F_* defines a differentiable functor!

It is now that we will wave our hands and say the word "orbifold" everywhere. Let Q and Q' be orbifolds. A map $f\colon Q \to Q'$ defines a differentiable map of orbifolds when for any atlas groupoid $(\mathcal{G}_Y, Y)$ for Q', there exists an atlas groupoid $(\mathcal{G}_X, X)$ for Q and a differentiable functor $F_*\colon \mathcal{G}_X \to \mathcal{G}_Y$ such that the resulting quotient map $\mathcal{G}\backslash F_*\colon \mathcal{G}_X\backslash X \to \mathcal{G}_Y\backslash Y$ is equal to $f\colon Q \to Q'$. Two such differentiable functors define the same map of orbifolds if they share a common refinement.

In other words, a map $f\colon Q \to Q'$ is a differentiable map of orbifolds whenever for any uniformizing chart $p_j\colon Y_j \to Q'$, and any open cover $(U_i)_{i\in I}$ of

$f^{-1}(p_j(Y_j))$ with uniformizing charts $q_i\colon X_i \to U_i$, we have differentiable maps $f_{ij}\colon X_i \to Y_j$ such that the following diagram commutes

$$\begin{array}{ccc} X_i & \xrightarrow{f_{ij}} & Y_j \\ \Big\downarrow{\scriptstyle q_i} & & \Big\downarrow{\scriptstyle p_j} \\ Q & \xrightarrow{f} & Q'. \end{array}$$

In particular, there exists a homomorphism (not really uniquely determined) $\hat{f}_{ij}\colon \Gamma_i \to \Gamma_j$ such that $f_{ij}(\gamma.x) = \hat{f}_{ij}(\gamma).f_{ij}(x)$ for all $\gamma \in \Gamma_i$.

For the moment, we will beg to be allowed to be a little cagey about the question of whether a differentiable map of orbifolds is uniquely determined by its expression as $f\colon Q \to Q'$. A differentiable map is a *diffeomorphism* if it has a two-sided inverse.

Covering Maps Let $p\colon O \to Q$ be a morphism of orbifolds. If there exists an atlas Y for O and an atlas X for Q such that the resulting map $P\colon Y \to X$ is a covering map, then we say p itself is a *(orbifold) covering map.*

By refining Y if necessary, we may make the following useful assumption: for every change of charts $h\colon U \to U'$ belonging to $\mathcal{H}$, there exists a change of charts $\hat{h}\colon P^{-1}(U) \to P^{-1}(U')$ such that

$$\begin{array}{ccc} P^{-1}(U) & \xrightarrow{\hat{h}} & P^{-1}(U') \\ \Big\downarrow{\scriptstyle P} & & \Big\downarrow{\scriptstyle P} \\ U & \xrightarrow{\quad h \quad} & U'. \end{array}$$

Furthermore, we have

$$\widehat{hh'} = \hat{h}\hat{h}' \text{ and } \widehat{h^{-1}} = \hat{h}^{-1}.$$

Indeed, we may define a pseudogroup of changes of charts for Y by taking the pseudogroup "generated by" the $\hat{h}$ (and their unions and restrictions to open sets). By passing to germs, we recover the action of P on $\mathcal{G}_Y$. Also, given a component X_i with associated finite group Γ_i, looking at the elements $\hat{\gamma}$ for $\gamma \in \Gamma_i$, we get an action of Γ_i on $P^{-1}(X_i)$. By choosing a component of the preimage Y_i, we get an injection of its stabilizer under the action of Γ_i, which we write as $\Gamma_{Y,i} \to \Gamma_i$. Alternatively, one could interpret this as saying that given a point $y \in Y$ with $P(y) = x$, P induces a monomorphism on isotropy groups $\mathcal{G}_y \to \mathcal{G}_x$.

Let x be a point of X. Suppose the cardinality of the set of $\mathcal{H}_Y$-orbits in $P^{-1}(X)$ is finite, and let $\{x_1, \dots, x_k\}$ be a set of representatives of these preimages. We have

$$\sum_{i=1}^{k} \#(\mathcal{G}_{x_i}) = d_x \cdot \#(\mathcal{G}_x).$$

In fact, if Q is connected, then as x varies, the quantity on the right-hand side stays constant. This is the *degree* of the cover. If one $\mathcal{H}_Y$-orbit is infinite, then all orbits are infinite, and we say the degree of the cover is infinite.

Galois Coverings Suppose, in the situation above, that there exists a group G such that G acts simply transitively on each fiber of P, i.e. that $P\colon Y \to X$ is a Galois cover with Galois group G, and suppose further that each lift $\hat{h}$ commutes with the G-action, in the sense that $\hat{h}(g.y) = g.\hat{h}(y)$. Notice that because each X_i is simply connected, the cover restricted to X_i is trivial. That is, given a section $X_i \to P^{-1}(X_i)$, we may identify $P^{-1}(X_i)$ with $G \times X_i$. Furthermore the G-action is by left-translation in the first factor and trivial in the second factor. Given $\gamma \in \Gamma_i$, we know that $\hat{\gamma}$ preserves $P^{-1}(X_i)$, and so we have

$$\hat{\gamma}.(g, x) = (g\varphi_i(\gamma)^{-1}, \gamma.x).$$

It is not immediately obvious why the action on G should be given by right translation after applying a homomorphism $\varphi_i\colon \Gamma_i \to G$, but if we write $\varphi_i(\gamma)^{-1}$ for the action of γ applied to $1 \in G$, we have

$$\begin{tikzcd}
1 \arrow[r, mapsto] \arrow[d, mapsto] & \varphi_i(\gamma)^{-1} \arrow[d, mapsto] \\
g \arrow[r, mapsto] & g\varphi_i(\gamma)^{-1},
\end{tikzcd}$$

demonstrating that the action of γ is by right translation. It follows easily that φ_i actually defines a homomorphism $\varphi_i\colon \Gamma_i \to G$. The reason for the presence of the inverse is that right translation is naturally a right action, but we have assumed that Γ_i acted on the left; inverting is the standard method of converting left actions to right actions.

Note that the action of G on Y descends to an action of G on O and yields a homeomorphism $G\backslash O \cong Q$. Let $Y_i = \{1\} \times X_i$ in the identification of $G \times X_i$ with $P^{-1}(X_i)$. Each connected component of Y is of the form $g.Y_i$ for some $i \in I$ and $g \in G$. Let $\hat{q}\colon Y \to O$ be the union of the uniformizing charts. The finite group associated to Y_i is the kernel of $\varphi_i\colon \Gamma_i \to G$. Let $\Gamma_{i,g}$ be the finite group resulting from restricting elements $\hat{\gamma}$ for $\gamma \in \Gamma_i$ such that $\gamma \in \ker \varphi_i$ to gY_i. The uniformizing chart $g.\hat{q}_i$ induces a homeomorphism $\Gamma_{i,g}\backslash Y_i \cong g.\hat{q}_i(Y_i)$.

By definition, we have $O = \bigcup_{(i,g)\in I\times G} g.\hat{q}_i(Y_i)$. By definition, the subgroup $\varphi_i(\Gamma_i)$ leaves $\hat{q}_i(Y_i)$ invariant; in fact, we have $g.\hat{q}_i(Y_i) \cap \hat{q}_i(Y_i) \neq \varnothing$ if and only if g belongs to $\varphi_i(\Gamma_i)$. Thus the group G acts properly on O! If further each φ_i is injective, then each uniformizing chart is actually a manifold chart; in this situation O is itself a differentiable manifold and Q is the quotient of O by the proper action of G.

References

[BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.